Characterization of Mixtures from Exponential and Pareto Distributions Using Left Truncated Moments

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[Received April 3, 2006; Revised December 22, 2006; Accepted January 29, 2007 ]

Abstract

In this work, the concept of left truncated moments are used to characterize a mixtures from two exponential distribution as well as a mixture from two Pareto Pareto distribution, conditional variance are obtained. We obtained the results of Talwalker (1977) for identifying the exponential and power distributions as a special cases. The recurrence relation of the conditional moments are obtained.

Keywords and Phrases: Characterization, left truncated moments, mixture of two exponential distributions, mixture of two Pareto distributions, failure rate.

AMS Classification: 62G30

1 Introduction

Identifying distributions is an important topic in statistics, reliability studies. There are various tools used to characterize the probability distributions. Galambos and Kots (1978) and Ahmed (1991) have used the concept of left truncated moments to characterize some probability distributions like exponential, Pareto, Power, Gamma, negative binomial, beta, binomial and Poisson. Talwalker (1977) used the concept of right truncated moments to identify different distributions like exponential and power Distributions. Characterizations by right truncated moments are very important in practice. For example, in reliability studies some measuring devices may be unable
to record values greater than time t. the main object of this work is to identify a mixture from two exponential as well as a mixture from the Pareto distributions using left truncated moments. Nassar and Mohmoud (1985) characterized a mixture from two exponential distributions using the left truncated moments. A mixture from two generalized gamma distributions has been characterized by Yehia (1992) using left truncated moments. Fakhry (1993) used the concept of conditional variance to identify a mixture from two Weibull distributions. In sections 2 and 3 we extended the results of talwalker (1977) concerning power and exponential distributions to mixtures of exponential as well as a mixture from two Pareto distributions.

2 Mixture of Exponential Distribution

The probability distribution function of the mixtures of two exponential distributions given by

\[ f(x) = \lambda_1 \theta_1 e^{-\theta_1 x} + \lambda_2 \theta_2 e^{-\theta_2 x} \]  \hspace{1cm} (1)

The cumulative distribution function is

\[ f(x) = 1 - \lambda_1 e^{-\theta_1 x} - \lambda_2 e^{-\theta_2 x} \]  \hspace{1cm} (2)

The reliability function is given by

\[ R(x) = \lambda_1 e^{-\theta_1 x} + \lambda_2 e^{-\theta_2 x} \]  \hspace{1cm} (3)

where \( \lambda_1 + \lambda_2 = 1 \), \( \theta_1 \) and \( \theta_2 \) are parameters.

**Theorem (2.1):**

Let \( X \) be a continuous random variable with finite \( r \)th moment, then \( x \) is a mixtures from two exponential distributions with positive parameters \( \theta_1 \) and \( \theta_2 \) iff

\[ E(x/x > y) = y + \frac{\theta_1 + \theta_2}{\theta_1 \theta_2} - \frac{r(y)}{\theta_1 \theta_2} \]  \hspace{1cm} (4)

Multiplying (3) by \(-\theta_2\) then add to (1) we get

\[ \lambda_1 = \frac{\theta_2 R(y) - f(y)}{(\theta_2 - \theta_1) e^{-\theta_1 y}} \]  \hspace{1cm} (5)
Multiplying (3) by $-\theta_1$ then add to (1) we get

$$\lambda_2 = \frac{f(y) - \theta_1 R(y)}{(\theta_2 - \theta_1)e^{-\theta_2 y}}$$

(6)

Now

$$E(x/x > y) = \frac{\int y f(x)dx}{R(x)}$$

$$\int y f(x)dx = \lambda_1 \int e^{-\theta_1 y} dx + \lambda_2 \int e^{-\theta_2 y} dx$$

Integrating by parts and substituting for $\lambda_1$ and $\lambda_2$ from equations (5) and (6), we have

$$\int y f(x)dx = y R(y) - \frac{f(y)}{\theta_1 \theta_2} + \frac{(\theta_1 + \theta_2) R(y)}{\theta_1 \theta_2}$$

Differentiating both sides w.r.t. $y$ and using the fact that $R'(y) = -f(y)$, we get

$$R''(y) + (\theta_1 + \theta_2) R'(y) + \theta_1 \theta_2 R(y) = 0$$

which is a second order differential equation has a solution of

$$R(y) = Ae^{-\theta_1 y} + Be^{-\theta_2 y}$$

for some constants $A$ and $B$. Since $R(0) = 1$, one gets $A + B = 1$ and thus $X$ is a mixtures from two exponential distribution, and

$$E(x/x > y) = y + \frac{\theta_1 + \theta_2}{\theta_1 \theta_2} - \frac{r(y)}{\theta_1 \theta_2}$$

$$E(x^2/x > y) = \frac{\int x^2 f(x)dx}{R(y)}$$
\[
\int_\infty^y x^2 f(x) dx = \lambda_1 \theta_1 \int_\infty^y x^2 e^{-\theta_1 x} dx + \lambda_2 \theta_2 \int_\infty^y x^2 e^{-\theta_2 x} dx
\]

Integrating by parts twice and substituting for \( \lambda_1 \) and \( \lambda_2 \) from equations (5) and (6), we have

\[
E(x^2/x > y) = y^2 + 2y \left( \frac{\theta_1 + \theta_2 - r(y)}{\theta_1 \theta_2} \right) + 2 \left( \frac{\theta_1^2 + \theta_1 \theta_2 + \theta_2^2 - (\theta_1 + \theta_2) r(y)}{\theta_1^2 \theta_2^2} \right) \tag{7}
\]

The conditional variance can be obtained as

\[
Var(x/x > y) = E(x^2/x > y) - [E(x/x > y)]^2
\]

\[
Var(x/x > y) = \frac{\theta_1^2 + \theta_2^2 - r^2(x)}{\theta_1^2 \theta_2^2}
\]

where

\[
r(y) = \frac{f(y)}{R(y)}
\]

is the failure rate function.

\( \lambda_1 = 1, \lambda_2 = 0, \theta_1 = \theta \) and \( \theta_2 = 0 \), then

\[
\int_\infty^y xf(x) dx = \left( y + \frac{1}{\theta} \right) e^{-\theta y}
\]

i.e.

\[
E(x/x > y) = y + \frac{1}{\theta}
\]

This result is a special case obtained by Talwalker (1977) for characterizing the exponential distribution.
3 Mixtures of two Pareto Distribution

The probability distribution function of the mixtures of two Pareto distribution is given by

$$f(x) = \lambda_1 a_1 \left(1/x\right)^{(a_1+1)} + \lambda_2 a_2 \left(1/x\right)^{(a_2+1)}$$  \hspace{1cm} (8)

The reliability function is given by

$$R(x) = \lambda_1 \left(1/x\right)^{a_1} + \lambda_2 \left(1/x\right)^{a_2}$$  \hspace{1cm} (9)

where $\lambda_1 + \lambda_2 = 1$, $a_1$ and $a_2$ are parameters.

Solving (8) and (9) for $\lambda_1$ and $\lambda_2$, we get

$$\lambda_1 = \frac{f(x) - a_2 R(x)/x}{(a_1 - a_2)(1/x)^{a_1+1}}$$  \hspace{1cm} (10)

and

$$\lambda_2 = \frac{a_1 R(x)/x - f(x)}{(a_1 - a_2)(1/x)^{a_2+1}}$$  \hspace{1cm} (11)

**Theorem (3.1):**

$$E(x/x > y) = \frac{\int_y^\infty xf(x)dx}{R(x)}$$

$$\int_y^\infty xf(x)dx = \int_y^\infty \lambda_1 a_1 x(1/x)^{a_1+1}dx + \int_y^\infty \lambda_2 a_2 x(1/x)^{a_2+1}dx$$

integrating and eliminating $\lambda_1$ and $\lambda_2$, we have

$$\int_y^\infty xf(x)dx = \frac{y^2 f(y) - a_1 a_2 y R(y)}{(1 - a_1)(1 - a_2)}$$
or

\[(1 - a_1)(1 - a_2) \int_{y}^{\infty} xf(x)dx = y^2 f(y) - a_1 a_2 y R(y)\]

Differentiating both sides w.r.t. \(y\), we have

\[y^2 R''(y) + (a_1 + a_2 + 1)y R'(y) + a_1 a_2 R(y) = 0\]

Which is the second order homogenous differential equation and has a solution of

\[R(y) = A(1/y)^{a_1} + B(1/y)^{a_2}\]

Since \(R(1) = 1\), one gets \(A + B = 1\), and thus \(X\) is a mixtures of two Pareto distribution.

\[
E(x/x > y) = \frac{\int_{y}^{\infty} xf(x)dx}{R(y)}
\]

\[
E(x/x > y) = \frac{a_1 a_2 y - y^2 r(y)}{(1 - a_1)(1 - a_2)}
\] (12)

\[
E(x^2/x > y) = \frac{\int_{y}^{\infty} x^2 f(x)dx}{R(y)}
\]

\[
\int_{y}^{\infty} x^2 f(x)dx = \int_{y}^{\infty} \lambda_1 a_1 x^2 x^{-a_1-1}dx + \int_{y}^{\infty} \lambda_2 a_2 x^2 x^{-a_2-1}dx\]

\[= \frac{\lambda_1 a_1}{2 - a_1} y^{2-a_1} + \frac{\lambda_2 a_2}{2 - a_2} y^{2-a_2}\]

Substituting for \(\lambda_1\) and \(\lambda_2\) from equations (10) and (11) respectively we have
\[
\int_{y}^{\infty} x^2 f(x) \, dx = \frac{2y^3 f(x) - a_1 a_2 y^2 R(y)}{(2 - a_1)(2 - a_2)}
\]

and hence

\[
E(x^2 / x > y) = \frac{a_1 a_2 y^2 - 2y^3 r(y)}{(2 - a_1)(2 - a_2)}
\]  \hspace{1cm} (13)

**Theorem (3.2):**

\[
E(x^m / x > y) = \frac{a_1 a_2 y^m - my^{m+1} r(y)}{(a_1 - m)(a_2 - m)}
\]  \hspace{1cm} (14)

**Proof:**

\[
E(x^m / x > y) = \frac{\int_{y}^{\infty} x^m f(x) \, dx}{R(y)}
\]

\[
\int_{y}^{\infty} x^m f(x) \, dx = \int_{y}^{\infty} \lambda_1 a_1 x^m x^{-a_1 - 1} \, dx + \int_{y}^{\infty} \lambda_2 a_2 x^m x^{-a_2 - 1} \, dx
\]

\[
= \frac{\lambda_1 a_1}{m - a_1} y^{m - a_1} + \frac{\lambda_2 a_2}{m - a_2} y^{m - a_2}
\]

Substituting for \(\lambda_1\) and \(\lambda_2\) from equations (10) and (11) respectively we have

\[
E(x^m / x > y) = \frac{a_1 a_2 y^m - my^{m+1} r(y)}{(a_1 - m)(a_2 - m)}
\]

from (8) if \(\lambda_1 = 1, \lambda_2 = 0, a_1 = a\) and \(a_2 = 0\), then

\[
\int_{y}^{\infty} x f(x) \, dx = \frac{a}{1 - a} y^{-a+1}
\]
And

\[ E(x/x > y) = \frac{a}{1 - a} y \]

which is the result obtained by Talwalker (1977) for characterizing Pareto distribution. From equations (12) and (13) the recurrence relation of moments can be obtained as

\[ \frac{1}{y}(2 - a_1)(2 - a_2)E(x^2/x > y) - (1 - a_1)(1 - a_2)E(x/x > y) = -y^2r(y) \]

from (14) we have

\[ E(x^{m+1}/x > y) = \frac{a_1a_2y^{m+2} - (m + 1)y^{m+2}r(y)}{(m - a_1)(m - a_2)} \] (15)

From (14) and (15) we have the following recurrence equation

\[ \frac{1}{y}(m - a_1)(m - a_2)E(x^{m+1}/x > y) - (m - a_1)(m - a_2)E(x^m/x > y) = -y^{m+1}r(y) \]

References


